

*In memory of my dear friend Alex Chigogidze*

## ORDERING A SQUARE

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ABSTRACT. We identify a condition on  $X$  that guarantees that any finite power of  $X$  is homeomorphic to a subspace of a linearly ordered space.

### 1. INTRODUCTION

To start our discussion, let us agree on terminology. A linearly ordered space, or a LOTS, is one whose topology is generated by open intervals and open rays with respect to some linear order on  $X$ . A space homeomorphic to a subspace of a LOTS is called a generalized ordered space, or a GO space. It is known (see, for example, [2]), a Hausdorff space  $L$  is a GO space, if its topology can be generated by a collection of convex sets (not necessarily open) with respect to some linear order on  $L$ . Throughout the paper, we will also refer to LOTS and GO spaces as orderable and suborderable, respectively.

Having a topology compatible with an order is a delicate property that can be destroyed by many standard operations. Under favorable conditions, however, order-topology ties demonstrate a remarkable resistance to the product operation as demonstrated by zero-dimensional separable metric spaces. A step into a non-metrizable world quickly reveals that zero-dimensionality has to be coupled with very strong properties to achieve a desired resistance of orderability to the product operation. For example,  $\omega_1$  is a zero-dimensional LOTS with "cannot be better" local properties, while  $\omega_1 \times \omega_1$  has a rather rigid non-orderable structure. One can see that  $\omega_1^2$  is not sub-orderable by observing that it is not hereditarily normal. Another natural example is Sorgenfrey Line, which is known to be a GO-space. The square of the line is not a GO-space. One explanation is non-normality. Observe, however, that every countable power of the Sorgenfrey Line admits a continuous injection into the irrationals.

The goal of this paper is to identify a property that may serve as a characterization of spaces with (sub) orderable finite powers. In Theorem 2.4, we present a sufficient conditions that may turn out to be a necessary one. We then observe that the condition in Theorem 2.4 is a criterion in the scope of well ordered subspaces (Theorem 2.5). We do not know if Theorem 2.4 can be reversed without narrowing its scope.

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In notation and terminology, we will follow [1]. Even though we study ordered spaces, we will not need a notation for an open interval. Therefore, we reserve  $(a, b)$  to denote the ordered pair. All spaces are assumed Tychonoff. We say that a subset  $A \subset X$  separates distinct  $x, y \in X$  if  $A$  meets  $\{x, y\}$  by exactly one element. A family  $\mathcal{A}$  of subsets of  $X$  separates distinct  $x, y \in X$  if at least one member of the family separates  $x$  from  $y$ .

## 2. RESULTS

To formulate our main result we would like to introduce a property that can be extracted from many arguments leading to orderability of certain spaces.

**Definition of the  $P$ -number of  $X$ .** *The  $P$ -number of a space  $X$  is  $|X|$  if  $X$  is discrete. Otherwise, the  $P$ -number of  $X$  is the largest cardinal number  $\tau$  such that the intersection of any fewer than  $\tau$  open sets of  $X$  is open.*

Note that the  $P$ -number of  $X$  is well-defined. Indeed, if  $X$  is not discrete, then there exists an infinite set  $A$  and  $b \in \bar{A} \setminus A$ . Then the family  $\{X \setminus \{a\} : a \in A\}$  consists of open sets and has non-open intersection. Thus, the  $P$ -number of  $X$  is

$$\min\{|\mathcal{O}| : \mathcal{O} \text{ consists of open sets and has nonopen intersection}\}$$

Since the minimum is computed over a non-empty set, it exists. In particular, the  $P$ -number of any non-discrete metric space is  $\omega$ . The  $P$ -number of  $\{\omega_1\} \cup \{\alpha \in \omega_1 : \alpha \text{ is isolated}\}$  is  $\omega_1$ .

Note that if a space has an  $\omega_1$ -long convergent sequence and an  $\omega$ -long convergent sequence, then the square of the space is not suborderable because the product of these two sequences is not hereditarily normal. This and similar structures are eliminated if a space has a  $\tau$ -discrete basis, where  $\tau$  is the  $P$ -number of the space. We start by identifying a necessary condition for suborderability. We then show that the found property is finitely productive.

**Theorem 2.1.** *Let  $X$  have a  $\tau$ -discrete basis of clopen sets, where  $\tau$  is the  $P$ -number of  $X$ . Then  $X$  is a GO-space.*

*Proof.* Let  $\mathcal{B} = \bigcup_{\alpha < \tau} \mathcal{B}_\alpha$  be a basis as in the hypothesis. Since each  $\mathcal{B}_\alpha$  is a discrete family of clopen sets, we may assume that  $\mathcal{B}_\alpha$  is a cover of  $X$  for each  $\alpha$ . Additionally, we may assume that  $\tau$  is infinite and  $\mathcal{B}_0 = \{X\}$ .

Inductively, for each  $\alpha < \tau$ , we will define a relation  $\mathcal{O}_\alpha$  on  $X$  so that  $\bigcup_{\alpha < \tau} \mathcal{O}_\alpha$  will be an order  $\prec$  on  $X$  compatible with the topology of  $X$ . In addition, we will define a collection  $\mathcal{P}_\alpha$  so that  $\bigcup_{\alpha < \tau} \mathcal{P}_\alpha$  will consist of  $\prec$ -convex sets and form a basis for the topology of  $X$ .

Step 0. Put  $\mathcal{O}_0 = \emptyset$  and  $\mathcal{P}_0 = \{X\}$ .

Assumption ( $\beta < \alpha$ ). Assume that for each  $\beta < \alpha$ , where  $\alpha < \tau$ , we have defined  $\mathcal{O}_\beta$  and the following hold:

- A1:  $\mathcal{O}_\gamma \subset \mathcal{O}_\beta$  if  $\gamma < \beta$ .
- A2: If  $x, y$  are separated by  $\bigcup_{\gamma \leq \beta} \mathcal{B}_\gamma$ , then either  $(x, y)$  or  $(y, x)$ , but not both, is in  $\mathcal{O}_\beta$ .
- A3: If  $(x, y), (y, z) \in \mathcal{O}_\beta$ , then  $(x, z) \in \mathcal{O}_\beta$ .
- A4:  $(x, x) \notin \mathcal{O}_\beta$  for any  $x \in X$ .
- A5: If  $x, y$  are not separated by any element of  $\bigcup_{\gamma \leq \beta} \mathcal{B}_\gamma$ , then the following two statements are true:

$$\forall z[(z, x) \in \mathcal{O}_\beta \rightarrow (z, y) \in \mathcal{O}_\beta]$$

$$\forall z[(x, z) \in \mathcal{O}_\beta \rightarrow (y, z) \in \mathcal{O}_\beta]$$

Note that our assumptions hold for  $\beta = 0$ .

Step  $\alpha < \tau$ . Put  $\mathcal{F}_\alpha = \bigcup_{\beta < \alpha} \mathcal{B}_\beta$ .

*Definition of  $\mathcal{P}_\alpha$ :*  $P \in \mathcal{P}_\alpha$  if and only if  $P$  is the non-empty intersection of a maximal subfamily of  $\mathcal{F}_\alpha$  with the finite intersection property.

The next three claims will be used in our post-construction argument.

*Claim 1.* Any  $I \in \mathcal{P}_\alpha$  is open in  $X$ .

Let  $\mathcal{I} \subset \mathcal{F}_\alpha$  be such that  $I = \bigcap \mathcal{I}$ . Since each  $\mathcal{B}_\beta$  is discrete,  $\mathcal{I}$  meets each  $\mathcal{B}_\beta$  for  $\beta < \alpha$  by exactly one element. Therefore,  $|\mathcal{I}| \leq \alpha < \tau$ . Since the  $P$ -number of  $X$  is  $\tau$ , the set  $I = \bigcap \mathcal{I}$  is open, which proves the claim.

*Claim 2.* If  $x \in B_x \in \mathcal{F}_\alpha$ , then  $x \in P \subset B_x$  for some  $P \in \mathcal{P}_\alpha$ .

Since each  $\mathcal{B}_\beta$  is a disjoint cover of  $X$ , the family  $\{B_{x,\beta} : x \in B_{x,\beta} \in \mathcal{B}_\beta, \beta < \alpha\}$  is a maximal subfamily in  $\mathcal{F}_\alpha$  with the finite intersection property and its intersection  $P$  has the desired properties, which completes the claim.

The next claim is obvious and is stated without a proof.

*Claim 3.*  $\mathcal{P}_\alpha$  is a partition of  $X$  inscribed in each  $\mathcal{B}_\beta$ ,  $\beta < \alpha$ .

Next order elements of  $\mathcal{B}_\alpha = \{B_{\alpha,\lambda} : \lambda < \tau_\alpha\}$ .

*Definition of  $\mathcal{O}_\alpha$ :* Put  $\mathcal{O}'_\alpha = \{(x, y) : x, y \in P \in \mathcal{P}_\alpha, x \in B_{\alpha,\lambda}, y \in B_{\alpha,\gamma}, \lambda < \gamma\}$  and  $\mathcal{O}_\alpha = \mathcal{O}'_\alpha \cup (\bigcup_{\beta < \alpha} \mathcal{O}_\beta)$ .

Let us verify A1-A5 for  $\alpha$ .

A1 check: This property follows from the fact that each  $\mathcal{O}_\beta$ , where  $\beta < \alpha$ , is represented in the union that defines  $\mathcal{O}_\alpha$ .

A2 check: Fix  $x, y$  separated by  $\bigcup_{\beta \leq \alpha} \mathcal{B}_\beta$ . If  $x, y$  are separated by  $B \in \mathcal{B}_\beta$  for some  $\beta < \alpha$ , then apply A2 for  $\beta$  and A1 for  $\alpha$ .

Otherwise, by Claim 3, there exists  $P \in \mathcal{P}_\alpha$  that contains  $\{x, y\}$ . Since  $\mathcal{B}_\alpha$  is a disjoint cover of  $X$  that separates  $x$  and  $y$ , we conclude that  $x \in B_{\alpha, \lambda}$  and  $y \in B_{\alpha, \gamma}$  for some  $\lambda \neq \gamma$ . By the definition of  $\mathcal{O}'_\alpha$ , either  $(x, y)$  or  $(y, x)$ , but not both, is in  $\mathcal{O}'_\alpha$  and, therefore, in  $\mathcal{O}_\alpha$ .

A3 check: Fix  $(x, y), (y, z) \in \mathcal{O}_\alpha$ .

Assume that both  $(x, y), (y, z) \in \mathcal{O}_\beta$  for some  $\beta < \alpha$ . Then  $(x, z) \in \mathcal{O}_\beta$  by A3 for  $\beta$ . Then  $(x, z) \in \mathcal{O}_\alpha$  by A1 for  $\alpha$ .

Assume that  $(x, y) \in \mathcal{O}_\beta$  and  $(y, z) \notin \mathcal{O}_\beta$  for some  $\beta < \alpha$ . By A2 for  $\alpha$ , we conclude that  $(y, z) \notin \mathcal{O}_\beta$ . Then  $y$  and  $z$  are not separated by  $\bigcup_{\gamma \leq \beta} \mathcal{B}_\gamma$ . Then  $(x, z) \in \mathcal{O}_\beta$  by A5 for  $\beta$ . Therefore,  $(x, z) \in \mathcal{O}_\alpha$  by A1 for  $\alpha$ .

Assume that  $(x, y) \notin \mathcal{O}_\beta$  and  $(y, z) \in \mathcal{O}_\beta$  for some  $\beta < \alpha$ . Then the previous argument applies.

Assume that neither  $(x, y)$  nor  $(y, z)$  is  $\bigcup_{\beta < \alpha} \mathcal{O}_\beta$ . Then there exists  $P \in \mathcal{P}_\alpha$  such that  $\{x, y, z\} \subset P$ . Also, there exist  $\beta < \gamma < \lambda$  such that  $x \in B_{\alpha, \beta}, y \in B_{\alpha, \gamma}, z \in B_{\alpha, \lambda}$ . Since  $\beta < \lambda$ , we conclude that  $(x, z) \in \mathcal{O}'_\alpha \subset \mathcal{O}_\alpha$ .

A4 check: Fix any  $x \in X$ . By assumption,  $(x, x) \notin \bigcup_{\beta < \alpha} \mathcal{O}_\beta$ . By definition,  $(x, x) \notin \mathcal{O}'_\alpha$ . Therefore,  $(x, x) \notin \mathcal{O}_\alpha$ .

A5 check: Assume that  $x$  and  $y$  are not separated by  $\bigcup_{\gamma \leq \alpha} \mathcal{B}_\gamma$ . Then there exist  $P \in \mathcal{P}_\alpha$  such that  $\{x, y\} \subset P$  and  $\gamma$  such that  $x, y \in B_{\alpha, \gamma} \in \mathcal{B}_\alpha$ . We will consider the case  $(z, x) \in \mathcal{O}_\alpha$ .

Assume  $(z, x) \in \mathcal{O}_\beta$  for some  $\beta < \alpha$ . Then  $(z, y) \in \mathcal{O}_\beta$  by A5 for  $\beta$ . By A1 for  $\alpha$ , we conclude that  $(z, y) \in \mathcal{O}_\alpha$ .

Assume  $(z, x) \notin \mathcal{O}_\beta$  for any  $\beta < \alpha$ . Then there exists  $P \in \mathcal{P}_\alpha$  such that  $\{z, x, y\} \subset P$ . Since  $(z, x) \in \mathcal{O}_\alpha$  and  $x \in B_{\alpha, \gamma}$  there exists  $\lambda < \gamma$  such that  $z \in B_{\alpha, \lambda}$ . Since  $y \in B_{\alpha, \gamma}$  we conclude that  $(z, y) \in \mathcal{O}'_\alpha \subset \mathcal{O}_\alpha$ .

The inductive construction is complete.

Define  $\prec$  by letting  $x \prec y$  if and only if  $(x, y) \in \bigcup_{\alpha < \tau} \mathcal{O}_\alpha$ . Let us show that the relation is a linear order. Non-reflexivity follows from A4. Transitivity follows from A3 and A1. To check comparability, fix distinct  $x, y \in X$ . Since  $X$  is Hausdorff and  $\mathcal{B}$  is a basis for the topology of  $X$ , there exists  $\alpha < \tau$  such that  $x$  and  $y$  are separated by  $\mathcal{B}_\alpha$ . By A2, either  $(x, y)$  or  $(y, x)$  is in  $\mathcal{O}_\alpha$ .

The conclusion of the theorem follows from the next two claims.

*Claim 4.*  $\bigcup_{\alpha < \tau} \mathcal{P}_\alpha$  forms a basis for the topology of  $X$ .

To prove the claim, fix  $x \in X$  and  $B \in \mathcal{B}_\alpha$  containing  $x$ . By Claim 3, there exists  $P \in \mathcal{P}_{\alpha+1}$  such that  $x \in P \subseteq B$ . By Claim 1,  $P$  is open in  $X$ . The claim is proved.

*Claim 5.* Every element of  $\bigcup_{\alpha < \tau} \mathcal{P}_\alpha$  is convex with respect to  $\prec$ .

Fix  $P \in \mathcal{P}_\alpha$  and  $x, y \in P$  with  $x \prec y$ . Fix any  $z$  such that  $x \prec z \prec y$ . Since  $x, y \in P \in \mathcal{P}_\alpha$ , we conclude that  $x, y$  are not separated by  $\bigcup_{\beta < \alpha} \mathcal{B}_\beta$ . If  $x, z$  were separated by  $\bigcup_{\beta < \alpha} \mathcal{B}_\beta$  then  $(x, z)$  would have been in  $\mathcal{O}_\beta$  for  $\beta < \alpha$ . By A5,  $(y, z)$  would have been in  $\mathcal{O}_\beta$ , contradicting  $z \prec y$ . Therefore,  $x, y, z$  are not separated by  $\bigcup_{\beta < \alpha} \mathcal{B}_\beta$ . By Claim 3,  $x, y, z \in P$ , which proves convexity of  $P$ .  $\square$

To state the promised necessary condition for suborderability of finite powers, we need the following lemma.

**Lemma 2.2.** *Let  $\tau$  be the  $P$ -number of non-discrete spaces  $X$  and  $Y$ . Let  $X$  and  $Y$  have  $\tau$ -discrete bases of clopen sets. Then the  $P$ -number of  $X \times Y$  is  $\tau$ , and  $X \times Y$  has a  $\tau$ -discrete basis of clopen sets.*

*Proof.* Let  $\{O_\alpha : \alpha < \kappa\}$  be a collection of open sets in  $X \times Y$ , where  $\kappa < \tau$ . Assume that  $S = \bigcap_{\alpha < \kappa} O_\alpha$  is not empty. Fix any  $(x, y) \in S$ . Then for each  $\alpha < \kappa$ , there exist  $U_\alpha, V_\alpha$  open neighborhoods of  $x$  and  $y$ , respectively, such that  $U_\alpha \times V_\alpha \subset O_\alpha$ . Since the  $P$ -number of  $X$  and  $Y$  is  $\tau$ , we conclude that  $U = \bigcap_{\alpha < \kappa} U_\alpha$  and  $V = \bigcap_{\alpha < \kappa} V_\alpha$  are open. Therefore,  $(x, y) \in U \times V \subset S$ . Hence  $S$  is open. Therefore, the  $P$ -number of  $X \times Y$  is greater than or equal to  $\tau$ . Since  $X \times Y$  contains a copy of  $X$ , the  $P$ -number of  $X \times Y$  is at most  $\tau$ . Thus, the  $P$ -number of  $X \times Y$  is  $\tau$ .

Next, let  $\mathcal{U}_\alpha$  be a discrete collection of clopen subsets of  $X$  such that  $\bigcup_{\alpha < \tau} \mathcal{U}_\alpha$  is a basis for the topology of  $X$ . Similarly, we fix a basis  $\bigcup_{\alpha < \tau} \mathcal{V}_\alpha$  for the topology of  $Y$ . Put  $\mathcal{B}_{\alpha\beta} = \{U \times V : U \in \mathcal{U}_\alpha, V \in \mathcal{V}_\beta\}$ . The family  $\mathcal{B}_{\alpha\beta}$  is a discrete collection of clopen sets, since  $\mathcal{U}_\alpha$  and  $\mathcal{V}_\beta$  are. By the definition of product topology,  $\bigcup_{\alpha, \beta < \tau} \mathcal{B}_{\alpha\beta}$  is a basis for the topology of  $X \times Y$ .  $\square$

The statement of the next Lemma is a simple corollary to Lemma 2.2

**Lemma 2.3.** *Let  $\tau$  be the  $P$ -number of a non-discrete  $X$ . Let  $X$  have a  $\tau$ -discrete basis of clopen sets. Then for any positive integer  $n$ , the  $P$ -number of  $X^n$  is  $\tau$  and  $X^n$  has a  $\tau$ -discrete basis of clopen sets.*

Theorem 2.1 and Lemma 2.3 imply our main result of the paper, stated as follows.

**Theorem 2.4.** *If the  $P$ -number of  $X$  is  $\tau$  and  $X$  has a  $\tau$ -discrete basis of clopen sets, then  $X^n$  is a generalized ordered space for any natural number  $n$ .*

Theorem 2.4 implies, in particular, that if  $X$  is zero-dimensional and the weight of  $X$  is equal to the  $P$ -number of  $X$ , then any finite power of  $X$  is suborderable. Since the square of the Sorgenfrey Line is not suborderable, the weight- $P$ -number equality cannot be replaced by the density- $P$ -number equality. Also, having  $P$ -number equal to weight locally and uniformly is not sufficient either. The space of countable ordinals serves as a counterexample.

It is natural to wonder if our sufficient condition for suborderability of  $X^2$  is a characterization. Our next result shows that it may be, at least for a sufficiently large class of spaces. For the sake of our next result only, we say that  $X$  is character-homogeneous at points of a set  $S \subset X$  if  $\chi(x, X) = \chi(y, X)$  for all  $x, y \in S$ . We will use the theorem of Engelking and Lutzer that (see [2] or [3]) "*A  $GO$ -space  $X$  is paracompact if and only if no close subspace of  $X$  is homeomorphic to a closed subspace of a regular uncountable cardinal*".

**Theorem 2.5.** *Let  $X$  be a subspace of an ordinal. Then the following conditions are equivalent:*

- (1)  *$X$  has no stationary subspaces and is character-homogeneous at all non-isolated points.*
- (2)  *$X$  has a  $\tau$ -discrete basis of clopen sets, where  $\tau$  is the  $P$ -number of  $X$ .*
- (3)  *$X^n$  is suborderable for any  $n$ .*

*Proof.* *Proof of (1) $\Rightarrow$ (2).* We will prove the implication by induction on ordinal  $\alpha$  that can host  $X$ .

*Step ( $n$  is finite).* Then  $X = \{a_1, \dots, a_m\}$  for some  $m \leq n$  and the  $P$ -number of  $X$  is  $m$ . The family  $\bigcup_{i \leq m} \mathcal{B}_i$ , where  $\mathcal{B}_i = \{\{x_n\} : n = 1, \dots, m\}$ , is an  $m$ -discrete basis of  $X$ .

*Remark.* Observe that a discrete space of any cardinality has a  $\tau$ -discrete basis for any  $\tau > 0$ ,

*Assumption for  $\beta < \alpha$ .* Assume that for any  $X \subset \beta < \alpha$ , the implication (1) $\Rightarrow$ (2) is true.

*Inductive Step  $\alpha$ .*

Case of limit  $\alpha$ . Since  $X$  has no subspaces homeomorphic to a stationary subset, by the Engelking-Lutzer Theorem,  $X$  is paracompact. Therefore,  $X$  can be written as a free sum  $\oplus_{\gamma \in \Gamma} X_\gamma$ , where  $X_\gamma \subset \gamma < \alpha$  for each  $\gamma \in \Gamma$ . Since  $X$  is character homogeneous at all non-isolated points, there exists a cardinal  $\tau$  such that  $\chi(x, X) = \tau$  for all non-isolated  $x \in X$ . Then each  $X_\gamma$  has no subspaces homeomorphic to a stationary subset and is character homogeneous at non-isolated points. The  $P$ -number of any non-discrete  $X_\gamma$  is  $\tau$ . By Inductive Assumption and Remark, each  $X_\gamma$  has a  $\tau$ -discrete basis  $\mathcal{B}_\gamma$  of clopen sets. Since  $\{X_\gamma : \gamma \in \Gamma\}$  is a discrete cover, the family  $\bigcup_{\gamma \in \Gamma} \mathcal{B}_\gamma$  is a  $\tau$ -discrete basis of  $X$  consisting of clopen sets.

Case of isolated  $\alpha$ . Let  $\alpha = \beta + 1$ . We may assume that  $\beta \in X$  and  $\beta$  is not isolated in  $X$ . Otherwise, we can reduce our scenario to a smaller ordinal.

Since  $X$  has no subspaces homeomorphic to a stationary subset, we conclude that  $X \setminus \{\beta\}$  can be written as a free sum  $\oplus_{\gamma \in \Gamma} X_\gamma$ , where  $X_\gamma \subset \gamma < \beta$  for each  $\gamma \in \Gamma$  and  $|\Gamma| = cf(\beta)$ . Since  $\beta$  is not isolated in  $X$ , we conclude that the character of  $\beta$  is  $cf(\beta)$ . Then the character at all non-isolated points of  $X$  is  $cf(\beta)$ . Following the argument of the limit case, each  $X_\gamma$  has a  $cf(\beta)$ -discrete basis of clopen sets  $\mathcal{B}_\gamma$ . The family

$$\{X \setminus (\gamma + 1) : \gamma \in \Gamma\} \cup \{B \in \mathcal{B}_\gamma : \gamma \in \Gamma\}$$

is  $cf(\beta)$ -discrete basis of  $X$  consisting of clopen sets.

*Proof of (2)  $\Rightarrow$  (3).* This implication is Theorem 2.4.

*Proof of (3)  $\Rightarrow$  (1).* If  $X$  had a stationary subset or two limit points of distinct characters then  $X \times X$  would not have been hereditarily normal. This statement follows from the argument of Katětov [4] that if  $X \times Y$  is hereditarily normal then either every closed subset of  $X$  is a  $G_\delta$ -set or every countable subset of  $Y$  is closed. Since every GO space is hereditarily normal, the implication is proved.  $\square$

We would like to finish the paper with a few questions that naturally arise as a result of our discussion.

**Question 2.6.** *Let  $X \times X$  be suborderable. Is  $X \times X$  orderable? What if  $X$  is orderable?*

**Question 2.7.** *Let  $X \times X$  be orderable (suborderable). Is  $X^n$  orderable (suborderable)?*

**Question 2.8.** *Let  $X$  have a  $\tau$ -discrete basis of clopen sets, where  $\tau$  is the  $P$ -number of  $X$ . Is  $X \times X$  orderable? What if  $\tau$  is the weight of  $X$ ?*

Finally, the unaccomplished goals of the paper are summarized in the next two questions.

**Question 2.9.** *Assume that  $X \times X$  is suborderable (or orderable). Is it true that  $X$  has a  $\tau$ -discrete basis of clopen sets, where  $\tau$  is the  $P$ -number of  $X$ ?*

**Question 2.10.** *Assume that  $X \times X$  is suborderable (or orderable) space of density  $\tau$ . Is it true that the weight of  $X$  is equal to the  $P$ -number of  $X$ ?*

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